of their results with Barrett's observations is not possible, as for the short wave-lengths used by Barrett "relativity effects" are appreciable, whereas they are neglected in the derivation of the theoretical formula. Owing to the factor introduced to correct for the change of wave-length, equation (2) shows that the quantity R as defined by Waller and Hartree is not a function of $\sin\frac{\theta}{2}/\lambda$ only, a result not in agreement with the conclusion drawn by these authors. A comparison of Waller and Hartree's results for argon with those of the present calculation also indicates that the absolute values of the intensity of scattering per electron for large angles of scattering given by these authors are much higher than those calculated according to Equation (2). Unfortunately no experimental results are available to decide this point.

Finally it may be remarked that by the method outlined above the scattering by all monatomic gases and vapours can be approximately estimated. The writer has numerically evaluated the scattering from He, A, Ne, Kr, Na, K and Hg for Mok_α and CuK_α radiation and experiments are in progress to test these results.

A detailed account of this work will be published elsewhere.

IRROTATIONAL MOTION OF A COMPRESSIBLE INVISCID FLUID

BY H. BATEMAN

CALIFORNIA INSTITUTE OF TECHNOLOGY

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1. *Variational Principles.*—Let us assume that in the free two-dimensional irrotational motion of a compressible inviscid fluid the "density of mechanical energy" $p + {}^1/_2\, \rho\, q^2$ is an assigned differentiable function $f(\rho)$ of the density ρ of the fluid. The pressure p is then given by the equation

$$p = f(\rho) - ({}^1/_2)\,\rho q^2 = f(\rho) - ({}^1/_2)\rho\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] \qquad (1)$$

where ϕ is the velocity potential. In steady motion the pressure energy

$$P = \int\int p\,dxdy \qquad (2)$$

will be an extremum (sometimes a maximum) for certain allowable variations of ρ and ϕ when, among other conditions, the relations

$$O = f'(\rho) - {}^1/_2\, q^2 \qquad (3)$$

$$\rho \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \rho \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{4}$$

are satisfied, ψ being a function analogous to that introduced by Haar[1] in his treatment of two-dimensional variational problems. In the hydrodynamical problem ψ is the stream-function and when the function ϕ is assumed to have continuous second derivatives the elimination of ψ gives the equation of continuity

$$\frac{\partial}{\partial x}\left(\rho \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho \frac{\partial \phi}{\partial y}\right) = 0. \tag{5}$$

This equation and (3) together represent the differential equations of Euler and Lagrange for this particular variation problem. The variational problem itself is a simple modification of one used by Clebsch.[2]

To find when the variation problem is regular we write $\rho + \epsilon\tau$ in place of ρ and $\phi + \epsilon\chi$ in place of ϕ, ϵ being a small quantity. The coefficient of $^1/_2\epsilon^2$ in the new expression for P is then

$$\int\int dxdy \left\{ r^2 \left[f''(\rho) + \frac{1}{\rho}\left(\frac{\partial \phi}{\partial x}\right)^2 + \frac{1}{\rho}\left(\frac{\partial \phi}{\partial y}\right)^2 \right] - \rho \left[\frac{\partial \chi}{\partial x} + \frac{\tau}{\rho}\frac{\partial \phi}{\partial x}\right]^2 - \rho\left[\frac{\partial \chi}{\partial y} + \frac{\tau}{\rho}\frac{\partial \phi}{\partial y}\right]^2 \right\}.$$

Now when use is made of (3) the pressure is given by the equation

$$p = f(\rho) - \rho f'(\rho), \tag{6}$$

and if c denotes the "local velocity of sound" we have

$$c^2 = \frac{dp}{d\rho} = -\rho f''(\rho). \tag{7}$$

The coefficient of $^1/_2\epsilon^2$ in P is thus negative when $q^2 < c^2$ at all points of the region of integration. If, on the other hand, there are places where, in the extremal motion, $q^2 < c^2$ we cannot assert that the pressure energy is a maximum.

It should be noticed that equation (7) gives

$$\int \frac{dp}{\rho} = -f'(\rho), \tag{8}$$

and so we have the energy-equation

$$\int \frac{dp}{\rho} + {}^1/_2\, q^2 = 0. \tag{9}$$

This is in accordance with the well-known fact that the mechanical energy does not represent the whole energy, for thermal energy must be taken into consideration. Equations (4) may be derived from a variational principle of the simple two-dimensional type considered by Haar by expressing p in terms of q with the aid of equation (3). The variational problem is then to make $\delta I = 0$ where

$$I = \int\int G(q^2)dxdy \tag{10}$$

and

$$q^2 = \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2, \tag{11}$$

the function ϕ being given certain allowable variations. This variational problem has already been considered by the author[3] and has been shown to be regular when the extremal function is such that $q^2 < c^2$ at all points of the region of integration. It should be mentioned that

$$G'(q^2) = -\,{}^1/_2\rho. \quad G''(q^2) = -\frac{1}{4f''(\rho)}, \quad \frac{G'(q^2)}{G''(q^2)} = 2\rho f''(\rho) = -2c^2. \tag{12}$$

When the function $G(q^2)$ is given, the function $f(\rho)$ may be obtained by solving the differential equation

$$f(\rho) - \rho f'(\rho) = G[2f'(\rho)]. \tag{13}$$

This is an equation of Clairaut's type and its general solution is

$$f(\rho) = C\rho + G[2C] \tag{14}$$

where C is an arbitrary constant. The general solution is not, however, of much interest from the physical standpoint; it is the singular solution which is needed and this is obtained by eliminating C from the equations

$$\begin{aligned} f(\rho) &= C\rho + G[2C] \\ O &= \rho + 2G'[2C]. \end{aligned} \tag{15}$$

In particular, if

$$G(q^2) = A + \sqrt{(1 + q^2)},$$

where A is a constant, we find that

$$\begin{aligned} f(\rho) &= A - {}^1/_2(\rho + 1/\rho) \\ p &= f(\rho) - \rho f'(\rho) = A - 1/\rho. \end{aligned}$$

When equations (3) and (4) are satisfied the first variations δP and δI may be expressed in the form

$$\int\int \frac{\partial(\psi,\delta\phi)}{\partial(x,y)}\, dxdy$$

and may be replaced by a line integral

$$\int \delta\phi . d\psi$$

round the boundary of the region of integration. The line integral vanishes when ψ is constant on one part of the boundary and $\delta\phi = 0$ on the other part.

Equations (3) and (4) may also be obtained from a variational principle of type $\delta J = 0$, where

$$J = \int\int dxdy \left[f(\rho) + \frac{1}{2\rho}\left\{ \left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 \right\} \right], \tag{16}$$

and the quantities ρ and ψ are to be given certain allowable variations. When (3) and (4) are satisfied the expression for J can be written in the form

$$J = \int\int dxdy \left[p + \frac{\partial(\phi,\psi)}{\partial(x,y)} \right], \tag{17}$$

and may be transformed into

$$P - \int \phi d\psi, \tag{18}$$

where the line integral is taken round the boundary of the region of integration. Thus J does not generally represent the pressure energy but it may do when the line integral vanishes on account of the assigned boundary conditions.

The integral which is to be varied may be expressed in another form by writing Q for the density of momentum, i.e.,

$$Q^2 = \left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 = \rho^2 q^2 = 2\rho^2 f'(\rho), \tag{19}$$

and using this equation to express the integrand in terms of Q alone. It should be noted that this equation may fail to give a real expression for ρ as a function of Q. A critical case occurs when the curves

$$Y = Q^2/2\rho^2, \quad Y = f'(\rho) \tag{20}$$

touch, that is, when

$$f''(\rho) = -Q^2/\rho^3 \tag{21}$$

or $c^2 = q^2$. When the equation does give real values of ρ these will generally be of two types, one for which $q^2 < c^2$ and another for which $q^2 > c^2$. When $q^2 = c^2$ the relations

$$q^2 = 2f'(\rho), \quad c^2 = -\rho f''(\rho) \tag{22}$$

indicate that c^2 has a special value a^2 and ρ a corresponding special value σ.

The partial differential equation for ψ may be written in the form

$$(Q^2 - c^2\rho^2)\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) = Q\left(\frac{\partial Q}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial Q}{\partial y}\frac{\partial\psi}{\partial y}\right). \tag{23}$$

from which it appears that in the critical case when $Q^2 = c^2\rho^2 = a^2\sigma^2$ the curve $Q^2 = a^2\sigma^2$ is orthogonal to the streamlines and is therefore an equipotential curve. An exceptional case arises, however, when Q^2 is a function of ψ only. The right-hand side of (23) then vanishes because Q^2 is a maximum for the special value of ψ which gives $Q^2 = a^2\sigma^2$.

When the two variational problems $\delta I = 0, \delta J = 0$ are both real and one is regular the other is also regular. In the particular case when $G(q^2) = (1 + q^2)^{1/2}$ the variational problem $\delta I = 0$ is equivalent to that of finding a minimal surface which satisfies certain boundary conditions. The associated variational problem $\delta J = 0$ is then specified by the function

$$H(Q)^2 = (1 - Q^2)^{1/2} \tag{24}$$

and the condition for regularity is easily seen to be $Q^2 < 1$, which is satisfied when the variational problem is real. It should be noticed that in this case

$$Q^2 = \frac{q^2}{1+q^2} \tag{25}$$

so although $\frac{\partial\phi}{\partial x}$ and $\frac{\partial\phi}{\partial y}$ may be allowed to have any real values in the original variation problem $\delta I = 0$, the derivatives in the variational problem $\delta J = 0$ are restricted by the condition $Q^2 < 1$.

When the variational problems $\delta I = 0, \delta J = 0$ are real and regular and the extremal functions ϕ and ψ have continuous second derivatives the partial differential equations for ϕ and ψ are of elliptic type. It then follows from a remark[4] made by Radó that the extremal functions satisfy the inequalities

$$\begin{aligned} \frac{\partial^2\phi}{\partial x^2}\,\frac{\partial^2\phi}{\partial y^2} &\leqq \left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2 \\ \frac{\partial^2\psi}{\partial x^2}\,\frac{\partial^2\psi}{\partial y^2} &\leqq \left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2. \end{aligned} \tag{26}$$

Indeed, if $Ar + 2Bs + Ct = 0$, where $AC > B^2$, we may write

$$4B^2(s^2 - rt) = (Ar + Ct)^2 - 4B^2rt$$

and it is easily seen that the quadratic form on the right is always positive. The surfaces represented by the equations $z = \phi(x,y), z = \psi(x,y)$ are thus saddle surfaces and we have the hydrodynamical theorem that when $q^2 < c^2$ throughout the region of integration in the variation problems we have the inequality

$$\frac{\partial(u, v)}{\partial(x, y)} \leqq 0 \tag{27}$$

at all points of this region.

Some other interesting properties of regular two-dimensional variation problems are given in Radó's paper.

Spiral Flow.—Q^2 is a function of ψ only in the case when ψ depends only on r, the distance from the origin. Using polar coördinates (r,θ) we proceed to study Taylor's solution[5]

$$\phi = A\theta + BF(r), \tag{28}$$

in which A and B are arbitrary constants and $F(r)$ is a function to be determined, taking the general case, when the function $f(\rho)$ is not restricted, to be of the adiabatic type. Noting that $\frac{\partial \phi}{\partial \theta} = A$ we see that q^2 depends only on r, hence ρ also depends only on r and the differential equation for ϕ takes the form

$$\rho\left[r\frac{\partial^2\phi}{\partial r^2} + \frac{\partial \phi}{\partial r}\right] + r\frac{\partial \rho}{\partial r}\frac{\partial \phi}{\partial r} = 0. \tag{29}$$

This may be integrated giving

$$\rho r\frac{\partial \phi}{\partial r} = \tau A \tag{30}$$

where τ is a constant. In Taylor's notation

$$\nu a^2 = \frac{1}{r^2}\left(\frac{\partial \phi}{\partial \theta}\right)^2, \quad \eta a^2 = \left(\frac{\partial \phi}{\partial r}\right)^2. \tag{31}$$

The relation takes the form

$$\rho^2\eta = \tau^2\nu \tag{32}$$

or

$$\nu a^2 = \frac{\rho^2 q^2}{\rho^2 + \tau^2}. \tag{33}$$

The solution of Taylor's differential equation

$$\frac{d\eta}{d\nu} = \frac{\eta}{\nu}\,\frac{\gamma + 1 + (3-\gamma)\nu - (\gamma-1)\eta}{\gamma + 1 - (\gamma - 1)\nu - (\gamma+1)\eta}, \tag{34}$$

which corresponds to the case in which

$$f'(\rho) = \frac{a^2}{2(\gamma - 1)}\left[\gamma + 1 - 2\,(\rho/\sigma)^{\gamma-1}\right], \tag{35}$$

where γ is a constant, is thus obtained by eliminating ρ from the equations

$$\nu a^2 = \frac{2\rho^2 f'(\rho)}{\rho^2 + \nu^2}$$
$$\eta a^2 = \frac{2\tau^2 f'(\rho)}{\rho^2 + \tau^2}. \tag{36}$$

It will be noticed that ν is a maximum when

$$\frac{2}{\rho} + \frac{f''(\rho)}{f'(\rho)} = \frac{2\rho}{\rho^2 + \tau^2}; \tag{37}$$

that is, when

$$\nu a^2 = 2f'(\rho) + \rho f''(\rho) = q^2 - c^2 \tag{38}$$

or $\eta a^2 = c^2$. The radial velocity is then equal to the local velocity of sound, thus Bickley's result (mentioned by Taylor) holds also in the general case. It appears that ρ is imaginary when r has a value less than that corresponding to the maximum value of ν and that when r has a value for which ρ can be real there are two possible values of ρ.

It should be noticed that when ν is a maximum $d\nu/d\rho$ is zero and $d\rho/d\nu$ infinite. Considered as a function of ν the density ρ has a branch point when ν is a maximum. For the corresponding value of r, then, two stream-lines corresponding to different functional expressions $\rho = R(r)$ come together and touch.

3. Motion with Velocity Greater Than the Local Velocity of Sound.—An interesting type of motion is obtained by writing

$$u = A(s),\ v = B(s),\ \rho = F(s). \tag{39}$$

The equation $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$ is then satisfied if

$$B'(s)\frac{\partial s}{\partial x} = A'(s)\frac{\partial s}{\partial y}, \tag{40}$$

and this equation may be satisfied by writing

$$xA'(s) + yB'(s) = C'(s), \tag{41}$$

where $C'(s)$ is an arbitrary continuous function. An appropriate expression for ϕ is now

$$\phi = xA(s) + yB(s) - C(s), \tag{42}$$

and the equation of continuity

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{43}$$

will be satisfied if the function $F(s)$ is chosen so that

$$F'(s)\,[A(s)A'(s) + (Bs)B'(s)] + F(s)\,[\{A'(s)\}^2 + \{B'(s)\}^2] = 0. \quad (44)$$

When the functions $A(s)$, $B(s)$, $C(s)$ are assigned the function $f(\rho)$ cannot be chosen arbitrarily. Indeed, if $f(\rho) = G(s)$ we have

$$f'(\rho)\ F'(s) = G'(s) \quad (45)$$

and since

$$2f'(\rho) = u^2 + v^2 = [A(s)]^2 + [B(s)]^2, \quad (46)$$

we have the equation

$$2G'(s) = F'(s)\ [\{A(s)\}^2 + \{B(s)\}^2] \quad (47)$$

for the determination of $G(s)$. When the function $G(s)$ has been found the pressure is given by the equation

$$p = G(s) - F(s)\,\frac{G'(s)}{F'(s)}, \quad (48)$$

while

$$C^2 = \frac{dp}{d\rho} = \frac{F(s)\ [F''(s)G'(s) - F'(s)G''(s)]}{[F'(s)]^3}$$

$$= \frac{[A(s)\,A'(s) + B(s)\,B'(s)]^2}{\{A'(s)\}^2 + \{B'(s)\}^2} \quad (49)$$

$$q^2 - c^2 = \frac{[A(s)\,B'(s) - A'(s)\,B(s)]^2}{\{A'(s)\}^2 + \{B'(s)\}^2} > 0. \quad (50)$$

The inequality shows that the present type of motion can exist only in a region where the fluid velocity is everywhere greater than the local velocity of sound. The motion has the interesting peculiarity that u, v, and ρ have constant values on each one of a certain family of straight lines given by the equation (41). An equipotential curve may be regarded as the envelope of a family of lines given by an equation of type (42). For each such family of lines there is a relation between p and characteristic of a fluid for which steady motion is possible with such a value of ϕ.

It should be noticed that in this type of motion we have

$$\frac{\partial(u, v)}{\partial(x, y)} = 0$$

and also

$$q/c = \mathrm{Sin}\ \xi$$

where Mach's angle ξ is the complement of the angle between the lines (41) and (42).

4. *The Circulation round an Airfoil.*—The analysis given by Glauert[6] is readily extended to the case in which the function $f(\rho)$ is of general form. Assuming that at a great distance r from the origin the component velocities u,v are approximately

$$u = V(\text{Cos}\,\theta + A/r), \quad v = -V(\text{Sin}\,\theta + B/r) \tag{51}$$

where A,B are functions of the polar angle θ and U is a constant, we have to this order

$$q^2/V^2 = 1 + 2(A\ \text{Cos}\,\theta + B\ \text{Sin}\,\theta)/r = 1 + 2\theta/r, \quad \text{say.} \tag{52}$$

We now assume that ρ differs only slightly from its value ρ_0 at infinity while p differs only slightly from the associated value p_0 and we write

$$C^2 = \frac{dp}{d\rho} = -\rho f''(\rho), \qquad a^2 \doteq -\rho_0 f''(\rho_0)$$

$${}^1/_2 q^2 = f'(\rho), \qquad {}^1/_2 V^2 = f'(\rho_0)$$

$$\frac{q^2}{U^2} = \frac{f'(\rho)}{f'(\rho_0)} = 1 + (\rho - \rho_0)f''(\rho_0)/f'(\rho_0)$$

$$= 1 + 2a^2(\rho_0 - \rho)/\rho_0 V^2 = 1 + 2(\rho_0 - \rho)/\lambda\rho_0,$$

where $\lambda = V^2/a^2$.

Therefore

$$\rho/\rho_0 = 1 - \lambda\Theta/r. \tag{53}$$

Also

$$p - p_0 = -\rho_0 f''(\rho_0)(\rho - \rho_0) = a^2(\rho - \rho_0)$$

$$= V^2(\rho - \rho_0)/\lambda = -\rho_0 V^2\,\Theta/r.$$

Therefore

$$\frac{r}{\rho}\frac{\partial\rho}{\partial r} = \frac{\lambda\Theta}{r}, \quad \frac{1}{\rho}\frac{\partial p}{\partial\theta} = -\rho_0\ V^2\Theta'/(r). \tag{54}$$

The analysis then proceeds as in Glauert's paper and it is found that

$$A = 0,\ 2\pi VB(1 - \lambda\ \text{Sin}^2\theta) = K\sqrt{(1 - \lambda)} \tag{55}$$

where U is the velocity of flow past the aerofoil and K is the circulation round it. Thus we have approximately

$$u = U \operatorname{Cos} \theta$$

$$v = - U \operatorname{Sin} \theta - \frac{K}{2\pi r} \frac{\sqrt{1-\lambda}}{1 - \lambda \operatorname{Sin}^2\theta} \tag{56}$$

and it may be proved as in Glauert's paper that when the above approximations are valid the effect of compressibility is to increase the lift in the ratio $1:\sqrt{1-\lambda}$, a result also given by Prandtl[7] and Ackeret.[8]

5. *The Effect of Varying the Function* $f(\rho)$.—Let us replace $f(\rho)$ by $f(\rho) + \epsilon F(\rho)$ where ϵ is so small that terms involving ϵ^2 may be neglected. The effect of this variation is to change ρ into $\rho + \epsilon\sigma$, ϕ into $\phi + \epsilon\chi$ and ψ into $\psi + \epsilon\omega$ in equations (2) and (3). The variational equations are thus

$$\begin{gathered} \sigma f''(\rho) + F'(\rho) = \frac{\partial\phi}{\partial x}\frac{\partial\chi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\chi}{\partial y} \\ \frac{\partial\omega}{\partial y} = \sigma\frac{\partial\phi}{\partial x} + \rho\frac{\partial\chi}{\partial x}, \quad -\frac{\partial\omega}{\partial x} = \sigma\frac{\partial\phi}{\partial y} + \rho\frac{\partial\chi}{\partial y}. \end{gathered} \tag{57}$$

In the case when the second derivatives of ϕ, χ and ω exist the differential equation for χ is

$$\frac{\partial}{\partial x}\left[\sigma\frac{\partial\phi}{\partial x} + \rho\frac{\partial\chi}{\partial x}\right] + \frac{\partial}{\partial y}\left[\sigma\frac{\partial\phi}{\partial y} + \rho\frac{\partial\chi}{\partial y}\right] = 0 \tag{58}$$

where σ is given by (57). The equation is of elliptic type when

$$\left[\rho f''(\rho) + \left(\frac{\partial\phi}{\partial x}\right)^2\right]\left[\rho f''(\rho) + \left(\frac{\partial\phi}{\partial y}\right)^2\right] > \left(\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\right)^2, \tag{59}$$

that is, when $c^2(c^2 - q^2) > 0$. The equation is thus of elliptic type when the differential equation for ϕ is of elliptic type.

[1] Haar, A., *Crelle*, **149**, 1 (1919); *Szeged Acta*, **3**, 224 (1927).

[2] Clebsch, A., *Ibid.*, **56**, 1 (1859).

[3] Bateman, H., *Proc. Roy. Soc. London*, **A125**, 598 (1929).

[4] Radó, T., *Szeged Acta*, **2**, 228 (1926).

[5] Taylor, G. I., *J. Lond. Math. Soc.*, **5**, 224 (1930).

[6] Glauert, H., *Proc. Roy. Soc. London*, **A118**, 113 (1928).

[7] Prandtl, L. See Ackeret's article, "Handbuch der Physik," Bd. 7, Ch. 5.

[8] Ackeret, J., *Helvetica Physica Acta*, **1**, 301 (1928).